AvivaColour

Displaced Diffusion LFM CEV Methodology

**France Life**

Version record

| Date | Version | Lead authors | Updated by | Change summary |
| --- | --- | --- | --- | --- |
| 07/04/2020 | V0.2 | Caleb Migosi |  | Initial draft |

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# Document Scope

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| --- | --- | --- |
| **Management BU** | France Life | |
| **Legal Entities** |  | |
|  | |
| **Valuation Basis** |  | |
| **Document Location** |  | |
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| **Peer Reviewer** |  | |
| **For Review** |  | |
| **Associated Documents** |  |  |

|  |  |  |
| --- | --- | --- |
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# Executive Summary

Adverse interest rate movements are a major risk concern to Aviva France’s portfolio. It is of particular importance to properly model and manage these risks. Model selection and calibration are therefore key to any interest rate risk management endeavour.

The anchor model for this paper is an extension of the *Lognormal Forward Model (LFM).* This is because the LFM can be calibrated on discretely discounted rates directly observable in the market.

The standard LFM, however, does not properly capture the volatility dynamics of complex financial instruments. It is for this reason that an extension of the model is used.

We expound on both the LFM and its extension, the Displaced Diffusion LFM Constant Elasticity of Variance (DD LFM CEV) and establish pricing and calibration parameters.

This will be done in the following key sections:

1. **Projection:**

* We define the forward diffusion under the spot Libor Measure.
* We also proceed to define an iterator allowing us to project zero coupon bonds.
* Finally, the section expounds on modelling the deflator.

1. **Swaption Pricing:** We provide the forward swap diffusion and the Chi-square closed form pricer.
2. **Calibration:**  We expound on the Levenberg Marquardt algorithm that will serve as the optimization function.

# Methodology

## Projection

The key particularity of the DD LFM CEV is the conditional volatility structure. We know anecdotally that implied volatility tends to move in the opposite direction to rates. This model allows us to replicate this relationship.

The DD LFM CEV also allows for the modelling of the forward swap rate under a forward rate framework using an industry standard approximation called the freezing technique and pricing of swaptions using a closed form formula. This allows us to price caps, floors and swaptions under a single model.

### Forward Diffusion under Spot LIBOR Measure

Two of the most basic interest rate market models are the **Lognormal Forward Model (LFM)** and the **Lognormal Forward-Swap (LSM) model.** The LFM has as its primary assumption that the forward rates are lognormally distributed. The LSM, on the other hand, has as its primary assumption that the swap rates are lognormally distributed. Our primary model of choice will be the LFM.

We define the forward rate at between as for a set of dates as:

where is the price of a zero coupon bond maturing at and is the time difference between and .

We define the diffusion of the forward rate under the spot Libor measure to be:

where is a 1-dimensional Brownian motion.

We also note that the component Brownian motions are independent i.e.

### Projection of Zero Coupons

We recall the definition of the forward rate and use a recurrence relation to find the value of our zero coupons.

The forward rate is defined as:

Therefore:

We can use this recursive definition to find the zero coupon price at :

### Projection of the Discount Factor

Seeing that the LFM is not a short rate model, particular attention should be given to the discount factor.

In the case of the LFM, the discount factor is modelled as the product of zero coupon bonds of the shortest possible maturity.

This implies defining the discount factor at time 0 to 1 i.e. .

We then recursively define:

### Discretization scheme

While choosing a discretization scheme, it is important to note that the parameter in the CEV precludes existence of negative values since it is less than one.

For this reason, the classical Euler scheme presented a number of implementation problems as the projected forward rate, , does in fact attain 0 for certain simulations.

We introduce a more exact discretization scheme presented by Hull & White (1999).

We define , the forward rate at time , , the time step and a standard Gaussian variable.

From the diffusion of the forward rate defined above, we use Ito’s lemma and define ) as to obtain an exact scheme defined as:

with and defined as in the diffusion.

We then deduce the forward rates from by defining:

## Swaption Pricing

In this section, we obtain the forward swap rate diffusion under the LFM framework. This methodology is derived from the standard LFM.

We use the already documented LFM CEV methodology to obtain a similar framework for the DD LFM CEV.

### Forward Swap Diffusion

In this section, we use the LFM model (defined in [Forward Diffusion under Spot LIBOR Measure](#Xc2702e0f0df91de129717d726cdf294b3e486c4)) to obtain the forward swap rate diffusion.

We divide this section into the following parts:

1. **Swap Dynamics:** We decompose swap rates as a combination of forward rates;
2. **Forward Swap Diffusion:** We define, under the decomposition in (1), a diffusion for the forward swap rate;
3. **Freezing Technique:** We apply the freezing technique to obtain an approximation of the diffusion.

#### Swap Dynamics

We begin with the valuation of an interest rate swap with a fixed rate , notional maturing at with representing the time difference between and . At initiation, the value of the floating leg is the par value and the value of the swap is 0. This implies that:

We can show that the value of is:

Under this definition, we can demonstrate that the forward swap initializing at and maturing at is:

We observe, however, that we can expand the numerator such that:

We multiply the numerator and denominator by and recall the definition of the forward to obtain:

Which we can simplify to:

#### Forward Swap Diffusion

###### Preliminary Note

Before going further, we note that this section is an application of the *Extended Market Model* framework by Andersen & Andreasen (1998) for volatility functions defined as:

The properties of the function are also elaborated in Section 3 of the paper.

###### From forward swap rate to forward swap diffusion

From the definition of the forward swap above and an application of Ito’s lemma, we can use the DD LFM CEV framework to define a diffusion for the forward swap under the forward swap measure:

where

We note that the term is largely intractable on account of the complexity of defining the drift term of the forward diffusion under the forward swap measure. (For more details, please refer to Brigo, Mercurio (2001).)

We can simplify our notation in the following manner: (t)

#### Freezing Technique

From the last section, we defined the diffusion of the swap under the DD LFM CEV framework to be:

We note that under the forward swap measure, the diffusion of the forward swap rate is a martingale. Using this fact, we can define the diffusion of the swap rate under, the forward swap measure:

We can multiply both the numerator and denominator by to obtain:

The freezing technique entails setting to. These weights are therefore frozen in time. Our final diffusion is, therefore an approximation of the diffusion under the CEV LSM framework allowing us to price both caps and swaptions.

The definitive approximation of the forward swap rate diffusion, therefore is:

### Swaptions Pricing (Chi-Square)

We have obtained a diffusion for the forward swap rate in the above section. In this section, we use this diffusion to arrive at an analytical pricing formula for the DD LFM CEV.

We divide this section into a number of parts:

1. **DD LFM CEV to Shifted CEV:** We obtain a unidimensional diffusion from the DD LFM CEV;
2. **Analytical Pricing:** We obtain an analytical formula for swaptions pricing;
3. **Hagan Approximation:** We obtain an approximative swaption pricing formula;
4. **Volatility Approximation:** In this section we obtain an approximation for the volatility term to be used in the closed form pricing formula;

#### DD LFM CEV to shifted CEV

We recall the final diffusion approximation:

We introduce a random time change defined as:

Drawing from Oksendal (2000)[[1]](#footnote-1), we can represent our multidimensional diffusion as:

where is a one dimensional Brownian motion.

We are therefore faced with two problems:

1. **Analytical pricing formula:** We require an analytical solution/approximation for the one dimensional shifted CEV.
2. **Volatility Approximation:** We also need to find the value/ approximation of the term.

The coming sections will address each of these problems.

#### Analytical Pricing Formula

A relationship between the CEV process and the non-central chi-square distribution was first established by Schroder (1998.)

We use a simple application of the Ito’s lemma to arrive at the same result for the displaced diffusion CEV.

We begin by recalling the shifted CEV process:

We apply a transformation of and apply both Ito’s Lemma and an interpretation of Feller’s classification[[2]](#footnote-2) to obtain the following analytical pricing formula for the swaption price:

is a non-central Chi Square cumulative distribution function whose parameters are:

#### Hagan Approximation

We have seen that it is possible to obtain a closed form pricing formula for swaptions under the CEV model. Operationally, however, the generation of the non-central Chi-square distribution may present challenges.

In light of this Hagan & Woodward (1999) provide a closed form formula allowing us to directly approximate the equivalent Black volatility from the term obtained in the previous section.

The details of the approximation have been provided in the Appendix[[3]](#footnote-3). The final approximation is:

Where is the strike and:

#### Volatility Approximation

From the formulas provided above, the most important missing parameter is the . Under the LSM, this parameter would have been readily available as the norm of the terms.

However, seeing that the LFM and LSM are incompatible, we are required to use an approximation.

In this section, we give a brief overview of the derivation of this approximation.

We begin with a definition by Andersen and Andreasen (1997) of volatility term as:

for the vector of *“frozen”* scalar weights and the matrix of all vector functions .

We can expand this expression to:

At this juncture it is important to note that is of the same dimension as the Brownian motions. We are, therefore, seeking the norm of a 2D vector, each dimension corresponding to the component Brownian motion.

Without loss of generality, we define the number of dimensions to be . It is also important to note that are the frozen weights: .

We therefore expand the inner product to obtain:

At this point, we can freeze the weight terms for the final expression:

This expression, however, can be simplified, by introducing a piecewise constant volatility structure which allows us to simplify the integral to a sum. This volatility structure will be discussed in the following section.

The sum under indices can also be simplified to a square operation. For t = 0, we obtain the following volatility approximation:

This is the **Hull & White approximation** and is one of the most commonly used volatility approximations in the market.

## Calibration

### Objective Function

We begin by recalling the volatility term (0):

We note the large number of parameters required in the calibration i.e. (all the terms along with their corresponding correlation structure.)

We can greatly reduce the number of parameters by introducing a parametrization of the volatility surface.

### Piecewise Volatility Parametrization

This structure also allows us to dissociate the calibration of the correlation structure of the forwards from the definition of the volatility function. This is particularly useful in reducing the dimensions for our objective function.

We introduce a piecewise constant parametrization of the following form:

where and .

To obtain the terms, we perform a PCA on the historical log ratio of the forward rates.

We begin by defining, a transformation of the forward rate:

Using this definition, we can, for each time step and maturity, approximate:

where is a vector of weights or loadings and the dimensions of the Browni

an motions used. The are components.

Noting the qth eigenvalue of the variance-covariance matrix of the log ratio of the forward rates, we obtain the following relation:

After normalization, we obtain the values of to be:

The correlation between any two forwards, is therefore defined as:

At this point, it is important to note that instead of directly calibrating the shift and conditionality terms , we will select a number of values and test the stability of our parameters.

From the above parametrization, we can define the set of our parameters, and our optimization problem:

for a loss function .

We note particularly that the market prices are not the actual prices quoted on the market but pseudo prices based on market implied volatilities and the EIOPA risk free curve.

### Optimization Algorithm (Levenberg Marquardt Algorithm)

In this section, we detail the Levenberg Marquardt Algorithm allowing us to minimize the following function:

In our case specifically, represents the parameter set and is a vector function whose result is the residual errors between the market price and calculated value from the model.

Explicitly, our functions are defined as:

The objective of the algorithm being to ensure that for a well chosen sequence , .

The basis of the component functions , therefore is:

where is the jacobian matrix at point x which can be approximated in the following manner:

for sufficiently small and a unit vector in the direction of j.

We note particularly that the sequence is constructed at each step (starting from an appropriate starting value ) such that:

The objective is to minimize the function:

The is the control term, penalising timesteps that are too big.

This allows us to find the solution that solves the equation:

#### Points to note

1. The algorithm interpolates between the Gauss Newton and Gradient Descent methods. The tuning parameter is large where the objective function requires rapid minimization in large steps. As the function approaches its minimum, the parameter decreases to give way to a quasi-Gauss Newton algorithm.
2. It is important not only to fix tolerance parameters but also exit procedures when the algorithm fails to converge.

# Annex

## Analytical Pricing Formula

We begin by recalling the shifted CEV process:

We can perform a change of variables and define the variable such that:

Applying Ito’s lemma, we can define as a square root process:

This equation is a squared Bessel process process with degress of freedom.

###### The Feller Classification

Feller(1951) studies diffusions of the class:

whose corresponding Fokker-Planck equation is:

where is the Dirac delta function.

Clearly, setting , and yields our BESQ equation.

###### Solution

The quantity is the probability that conditional on

After lengthy calculation elaborated in Brecher & Lindsay (2010), we obtain the value of a swaption to be:

where:

and:

We obtain the final result to be:

## Detailed Hagan Approximation

###### Singular Perturbation Methods

Singular perturbation theory provides a theoretical underpinning that allows us to approximate solutions to problems containing very small parameters that cannot be approximated by simply setting the values to 0. We call this small parameter.

We suppose that our problem function is . The primary presupposition of the singular perturbation theory is that this function can be expressed as:

In the case of specific differential equations, this is particularly interesting as expanding the function provides a system of differential equations that may be much easier to solve. Our problem is one such case

###### Generalized Model

This model generalizes the Black Model to allow for correct pricing of options across strikes and exercise dates without adjustment.

We define a general Black diffusion under the forward swap measure :

where is a deterministic function and is an Arrow-Debreu security.

The value of our swaption at date , therefore is:

We simplify this problem by denoting the expectation as a function of the time and forward value .

The expectation is defined on a probability distribution generated by the process

It therefore satisfies the backward Kolmogrov equation[[4]](#footnote-4):

with the condition

###### Application of Singular Perturbation Methods

At this point, we have a partial differential equation. We can therefore use singular perturbation methods to find the solution.

We do this by scaling the parameters and using SPM to arrive at an approximate solution.

We define our :

We also define:

We therefore replace the values of , , and with , and to obtain:

We consider the term and take its Taylor expansion;

We replace the value of with the expression above to obtain:

with the same boundary condition.

###### Asymptotic Expansion

We recall the importance of singular perturbation theory and the asymptotic expansion hypothesis behind the former. Using the above, we solve the equation by taking the asymptotic expansion of i.e.

We plug this into the main equation to obtain a system of differential equations:

We solve these first 3 orders to obtain the values of , and below[[5]](#footnote-5) :

This allows us to find the expression of :

We note that this expression is a Taylor expansion of on the term:

We denote this value and therefore:

###### Equivalent Black Volatility

From the previous expression, we can find the value of the swaption to be:

Recall, however, that and . We reapply the subsitutions to obtain:

We can apply a Taylor expansion on the expression to obtain:

Our final value of therefore is:

We plug this value back to our formula to obtain the swaption price.

We consider the case of the Black Model for the swap diffussion:

We replace with , with (since ).

Similarly, and . The value of ,therefore, is:

The price of the swaption under the Black Model, therefore is:

To obtain the equivalent Black volatility, we solve for by equating to :

We obtain as our :

###### CEV Application

We recall our original forward swap rate model:

We denote and . We then replace these values to find the equivalent Black volatility :

where:

We input this into the Black pricer to obtain the value of the swaption.

1. cf. Section 3.2.2.1 of Oksendal (2000) [↑](#footnote-ref-1)
2. Cf. [Analytical Pricing Formula](#_Analytical_Pricing_Formula) [↑](#footnote-ref-2)
3. Cf. [Detailed Hagan Approximation](#_Detailed_Hagan_Approximation) [↑](#footnote-ref-3)
4. cf [Backward Kolmogrov Equation](#backward-kolmogrov-equation) [↑](#footnote-ref-4)
5. cf. [Solutions to Perturbation Equations](#solutions-to-perturbation-equations) [↑](#footnote-ref-5)